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Ermakov Approach for Minisuperspace Oscillators

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The Wheeler–DeWitt equation of arbitrary Hartle–Hawking factor ordering for several minisuperspace universe models, such as the pure gravity Friedmann–Robertson–Walker and Taub ones, is mapped onto the dynamics of corresponding classical oscillators. The latter ones are studied by the classical Ermakov invariant method, which is a natural approach in this context. For the more realistic case of a minimally coupled massive scalar field, one can study, within the same type of approach, the corresponding squeezing features as a possible means of describing cosmological evolution. Finally, we comment on the analogy with the accelerator physics.

The formalism of Ermakov-type invariants (Ermakov, 1880; Kaushal, 1998) can be a useful, alternative method of investigating evolutionary and chaotic dynamical problems in the "quantum" cosmological framework (Cotsakis *et al.*, 1998; Kim, 1996). Moreover, the method of adiabatic invariants is intimately related to geometrical angles and phases (Anandan *et al.*, 1997; Berry, 1984; Hannay, 1985; Shapere and Wilczek, 1989) so that one may think of cosmological Hannay's angles as well as various types of topological phases (Dutta, 1993a,b). In the following we apply the formal Ermakov scheme to some of the simplest cosmological pure gravity oscillators, such as the empty Friedmann–Robertson–Walker (EFRW) "quantum" universes and the anisotropic Taub ones. The EFRW Wheeler–DeWitt (WDW) minisuperspace equation reads (Moncrief and Ryan, 1991; Obregón and Socorro, 1996; Rosu, 1998)

$$\frac{d^2\Psi}{d\Omega^2} + Q\frac{d\Psi}{d\Omega} - \kappa e^{-4\Omega}\Psi(\Omega) = 0, \tag{1}$$

where Q, assumed nonzero (Rosu and Socorro, 1998) and kept as a free parameter, is the Hartle–Hawking (HH) parameter for the factor ordering (Hartle and Hawking, 1983), the variable Ω is Misner's time (Misner, 1969a,b), and κ

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is the curvature index of the FRW universe; $\kappa = 1, 0, -1$ for closed, flat, open universes, respectively. For $\kappa = \pm 1$ the general solution is expressed in terms of Bessel functions, $\Psi_\alpha(\Omega) = e^{-2\alpha\Omega}\ (C_1 I_\alpha(\frac{1}{2}e^{-2\Omega}) + C_2 K_\alpha(\frac{1}{2}e^{-2\Omega}))$ and $\Psi_\alpha(\Omega) = e^{-2\alpha\Omega}\ (C_1 J_\alpha(\frac{1}{2}e^{-2\Omega}) + C_2 Y_\alpha(\frac{1}{2}e^{-2\Omega}))$, respectively, where $\alpha = Q/4$. The case $\kappa = 0$ is special/degenerate, leads to hyperbolic functions and will not be dealt with here. Equation (1) can be mapped in a known way to the canonical equations for a classical point particle of mass $M = e^{Q\Omega}$, generalized coordinate $q = \Psi$, momentum $p = e^{Q\Omega}\dot{\Psi}$, (i.e., velocity $v = \dot{\Psi}$), and identifying Misner's time Ω with the classical Hamiltonian time. Thus, one is led to

$$\dot{q} \equiv \frac{dq}{d\Omega} = e^{-Q\Omega} p \tag{2}$$

$$\dot{p} \equiv \frac{dp}{d\Omega} = \kappa e^{(Q-4)\Omega} q. \tag{3}$$

These equations describe the canonical motion for a classical EFRW point universe as derived from the time-dependent Hamiltonian of the inverted oscillator type (Baskoutas and Jannussis, 1992)

$$H_{\rm cl}(\Omega) = e^{-Q\Omega}\frac{p^2}{2} - \kappa e^{(Q-4)\Omega}\frac{q^2}{2}. \tag{4}$$

For this classical EFRW Hamiltonian the triplet of phase-space functions $T_1 = \frac{p^2}{2}$, $T_2 = pq$, and $T_3 = \frac{q^2}{2}$ forms a dynamical Lie algebra (i.e., $H = \sum_n h_n(\Omega)$ $T_n(p, q)$), which is closed with respect to the Poisson bracket, or more exactly $\{T_1, T_2\} = -2T_1$, $\{T_2, T_3\} = -2T_3$, $\{T_1, T_3\} = -T_2$. Using this algebra $H_{\rm cl}$ reads

$$H_{\rm cl} = e^{-Q\Omega} T_1 - \kappa e^{(Q-4)\Omega} T_3. \tag{5}$$

The Ermakov invariant $\mathcal{I}$ belongs to the dynamical algebra, i.e., one can write $\mathcal{I} = \sum_r \epsilon_r(\Omega) T_r$, and by means of $\frac{\partial \mathcal{I}}{\partial \Omega} = -\{\mathcal{I}, H\}$ one is led to the following equations for the functions $\epsilon_r(\Omega)$:

$$\dot{\epsilon}_r + \sum_n \left[\sum_m C^r_{nm} h_m(\Omega)\right] \epsilon_n = 0, \tag{6}$$

where C^r_{nm} are the structure constants of the Lie algebra that have been already given above. Thus, we get

$$\begin{aligned} \dot{\epsilon}_1 &= -2e^{-Q\Omega}\epsilon_2 \\ \dot{\epsilon}_2 &= -\kappa e^{(Q-4)\Omega}\epsilon_1 - e^{-Q\Omega}\epsilon_3 \\ \dot{\epsilon}_3 &= -2\kappa e^{(Q-4)\Omega}\epsilon_2. \end{aligned} \tag{7}$$

The solution of this system can be readily obtained by setting $\epsilon_1 = \rho^2$ giving $\epsilon_2 = -e^{Q\Omega}\rho\dot{\rho}$ and $\epsilon_3 = e^{2Q\Omega}\dot{\rho}^2 + \frac{1}{\rho^2}$, where ρ is the solution of the Milne–Pinney

(MP) equation (Milne, 1930; Pinney, 1950), $\ddot{\rho} + Q\dot{\rho} - \kappa e^{-4\Omega}\rho = \frac{e^{-2Q\Omega}}{\rho^3}$. There is a well-defined prescription going back to Pinney's note in 1950 of writing ρ as a function of the particular solutions of the corresponding parametric oscillator problem, i.e., the modified Bessel functions in the EFRW case. In the formulas herein, we shall keep the symbol ρ for this known function. In terms of the function $\rho(\Omega)$ the Ermakov invariant reads (Lewis, 1968)

$$\mathcal{I}_{\mathrm{EFRW}} = \frac{(\rho p - e^{Q\Omega}\dot{\rho}q)^2}{2} + \frac{q^2}{2\rho^2} = \frac{e^{2Q\Omega}}{2}(\rho\dot{\Psi}_\alpha - \dot{\rho}\Psi_\alpha)^2 + \frac{1}{2}\left(\frac{\Psi_\alpha}{\rho}\right)^2. \quad (8)$$

Next, we calculate the time-dependent generating function allowing one to pass to new canonical variables for which $\mathcal{I}$ is chosen to be the new "momentum" $S(q, P = \mathcal{I}, \vec{\epsilon}(\Omega)) = \int^q dq' p(q', \mathcal{I}, \vec{\epsilon}(\Omega))$ leading to

$$S(q, \mathcal{I}, \vec{\epsilon}(\Omega)) = e^{Q\Omega}\frac{q^2}{2}\frac{\dot{\rho}}{\rho} + \mathcal{I}\arcsin\left[\frac{q}{\sqrt{2\mathcal{I}\rho^2}}\right] + \frac{q\sqrt{2\mathcal{I}\rho^2 - q^2}}{2\rho^2}, \quad (9)$$

where we have put to zero the constant of integration. Thus, $\theta = \frac{\partial S}{\partial \mathcal{I}} = \arcsin(\frac{q}{\sqrt{2\mathcal{I}\rho^2}})$. Moreover, the canonical variables are now $q = \rho\sqrt{2\mathcal{I}}\sin\theta$ and $p = \frac{\sqrt{2\mathcal{I}}}{\rho}(\cos\theta + e^{Q\Omega}\dot{\rho}\rho\sin\theta)$. The dynamical angle will be $\Delta\theta^d = \int_{\Omega_0}^{\Omega}\langle\frac{\partial H_{\mathrm{new}}}{\partial \mathcal{I}}\rangle d\Omega' = \int_0^{\Omega}[\frac{e^{-Q\Omega'}}{\rho^2} - \frac{1}{2}\frac{d}{d\Omega'}(e^{Q\Omega'}\dot{\rho}\rho) + e^{Q\Omega'}\dot{\rho}^2]d\Omega'$, whereas the geometrical angle reads $\Delta\theta^g = \frac{1}{2}\int_{\Omega_0}^{\Omega}[\frac{d}{d\Omega'}(e^{Q\Omega'}\dot{\rho}\rho) - 2e^{Q\Omega'}\dot{\rho}^2]d\Omega'$. Thus, the total change of angle is $\Delta\theta = \int_{\Omega_0}^{\Omega}\frac{e^{-Q\Omega'}}{\rho^2}d\Omega'$. On the Misner time axis, going to $-\infty$ means going to the origin of the universe, whereas $\Omega_0 = 0$ means the present epoch. Using these cosmological limits we obtain *the interesting result* that the total change of angle $\Delta\theta$ during the cosmological evolution in Ω time can be written up to a sign as the Laplace transform of parameter Q of the inverse square of the MP function, $\Delta\theta = -L_{1/\rho^2}(Q)$.

We now briefly sketch the two-dimensional empty minisuperspace Taub model for which the Taub WDW equation reads

$$\frac{\partial^2\Psi}{\partial\Omega^2} - \frac{\partial^2\Psi}{\partial\beta^2} + Q\frac{\partial\Psi}{\partial\Omega} + e^{-4\Omega}V(\beta)\Psi = 0, \quad (10)$$

where $V(\beta) = \frac{1}{3}(e^{-8\beta} - 4e^{-2\beta})$. This equation can be separated in the variables $x_1 = -4\Omega - 8\beta$ and $x_2 = -4\Omega - 2\beta$. Thus, one gets the following two independent 1D problems for which the Ermakov procedure can be repeated along the lines of the EFRW case:

$$\frac{d^2\Psi_{T1}}{dx_1^2} + \frac{Q}{12}\frac{d\Psi_{T1}}{dx_1} + \left(\frac{\omega^2}{4} - \frac{1}{144}e^{x_1}\right)\Psi_{T1} = 0 \quad (11)$$

and

$$\frac{d^2\Psi_{T2}}{dx_2^2} - \frac{Q}{3}\frac{d\Psi_{T2}}{dx_2} + \left(\omega^2 - \frac{1}{9}e^{x_2}\right)\Psi_{T2} = 0. \tag{12}$$

The quantity $\omega/2$ is the separation constant, which is physically related to the wavenumber of a positive energy level in a Schroedinger interpretation. The solutions are $\Psi_{T1} \equiv \Psi_{T\alpha_1} = e^{(-Q/24)x_1} Z_{i\alpha_1}(ie^{x_1/2}/6)$ and $\Psi_{T2} \equiv \Psi_{T\alpha_2} = e^{(Q/6)x_2} Z_{i\alpha_2}(i2e^{x_2/2}/3)$, respectively, where $\alpha_1 = \sqrt{\omega^2 - (Q/12)^2}$ and $\alpha_2 = \sqrt{4\omega^2 - (Q/3)^2}$. The standard Ermakov procedure can be applied to each of the Eqs. (11) and (12).

A more realistic case is provided by the minimally coupled FRW massive-scalar-field minisuperspace model. The Ermakov approach, which differs from the previous one, will be studied in detail elsewhere. The WDW equation reads

$$\left[\partial_\Omega^2 + Q\partial_\Omega - \partial_\phi^2 - \kappa e^{-4\Omega} + m^2 e^{-6\Omega}\phi^2\right]\Psi(\Omega, \phi) = 0, \tag{13}$$

and can be written as a two-component Schroedinger equation (see e.g., Mostafazadeh, 1998). This allows one to think of cosmological squeezed states based on the Ermakov approach (Hartley and Ray, 1982; Pedrosa, 1987; Pedrosa and Bezerra, 1997). For this one makes use of the factorization of the Ermakov invariant $\mathcal{I} = \hbar(bb^\dagger + \frac{1}{2})$, where $b = (2\hbar)^{-1/2}[\frac{q}{\rho} + i(\rho p - e^{Q_c\Omega}\dot{\rho}q)]$ and $b^\dagger = (2\hbar)^{-1/2}[\frac{q}{\rho} - i(\rho p - e^{Q_c\Omega}\dot{\rho}q)]$. Q_c is a fixed HH factor ordering parameter. Let us now consider a reference Misner-time–independent oscillator with the Misner frequency ω_0 corresponds to an arbitrary epoch Ω_0 for which one can write the common factorizing operators $a = (2\hbar\omega_0)^{-1/2}[\omega_0 q + ip]$, $a^\dagger = (2\hbar\omega_0)^{-1/2}[\omega_0 q - ip]$. The connection between the a and b pairs is given by $b(\Omega) = \mu(\Omega)a + \nu(\Omega)a^\dagger$ and $b^\dagger(\Omega) = \mu^*(\Omega)a^\dagger + \nu^*(\Omega)a^\dagger$, where $\mu(\Omega) = (4\omega_0)^{-1/2}[\rho^{-1} - ie^{Q_c\Omega}\dot{\rho} + \omega_0\rho]$ and $\nu(\Omega) = (4\omega_0)^{-1/2}[\rho^{-1} - ie^{Q_c\Omega}\dot{\rho} - \omega_0\rho]$ fulfill the well-known relationship $|\mu(\Omega)|^2 - |\nu(\Omega)|^2 = 1$. The corresponding uncertainties are known to be $(\Delta q)^2 = \frac{\hbar}{2\omega_0}|\mu - \nu|^2$, $(\Delta p)^2 = \frac{\hbar\omega_0}{2}|\mu + \nu|^2$, and $(\Delta q)(\Delta p) = \frac{\hbar}{2}|\mu + \nu||\mu - \nu|$ showing that in general the Ermakov squeezed states are not minimum uncertainty states (Pedrosa, 1987; Pedrosa and Bezerra, 1997).

Finally, we recall that (Courant and Snyder, 1958; also see second footnote in Lewis (1976)), the Ermakov invariant is equivalent to the Courant–Snyder one in accelerator physics, which defines the admittance of the accelerating device. This allows in a certain sense a beam physics approach to cosmological evolution. The point is that under the assumption of no coupling between the radial and the vertical betatron oscillations, the latter ones are described by the Hill equation $z'' + n(s)\kappa_0^2 z = 0$, where n is the magnetic field index, and κ_o is the curvature of the orbit parametrized by s that may be considered as a counterpart of Ω. The solutions can be written as $z_\pm = w(s)e^{\pm i\psi(s)}$ and only a real ψ leads to bounded oscillations. The amplitude $w(s)$ satisfies an MP equation and moreover $\psi = w^{-2}$.

However, in "quantum" cosmology, ψ is a pure imaginary action functional leading to instabilities in the Ψ solutions. In other words, while in accelerators we are interested in stable periodic solutions, in "quantum" cosmology there are the unstable parametric solutions that come into play.

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